Inequality 13

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Q) $a_1, a_2, \dots, a_n \in \mathbb{R}^+$ and $\sum_{i=1}^{n} a_i = 1$, then prove that,

$$\sum_{i=1}^{n} \frac{a_i}{\sqrt{1-a_i}} \geq \frac{1}{\sqrt{n-1}} \sum_{i=1}^{n} \sqrt{a_i}$$

$AM \geq GM$

$\frac{1}{AM} \leq \frac{1}{GM}$

Ans:- $$\frac{a_i}{\sqrt{1-a_i}} = \frac{1}{\sqrt{1-a_i}} - \sqrt{1-a_i}$$

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{1-a_i}}\right) \geq \sqrt[n]{\prod_{i=1}^{n} \frac{1}{\sqrt{1-a_i}}} = \left(\prod_{i=1}^{n} \frac{1}{1-a_i}\right)^{\frac{1}{2n}} = \left(\frac{1}{\underbrace{\prod_{i=1}^{n}(1-a_i)}_{GM^n}}\right)^{\frac{1}{2n}}$$

as it is $\frac{1}{n}S_1$

$$\geq \left(\frac{1}{\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n}(1-a_i)\right)^n}_{AM^n}}\right)^{\frac{1}{2n}} = \left(\frac{n^n}{\left(n-\sum_{i=1}^{n}a_i\right)^n}\right)^{\frac{1}{2n}} = \left(\frac{n^n}{(n-1)^n}\right)^{\frac{1}{2n}} = \left(\frac{n}{n-1}\right)^{\frac{1}{2}}$$

$$S_1 \geq \left(\frac{n}{n-1}\right)^{\frac{1}{2}} n$$

$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$

$\Rightarrow \sum a_i b_i \leq \left(\left(\sum a_i^2\right)\left(\sum b_i^2\right)\right)^{\frac{1}{2}}$

$S = \sum \frac{a_i}{\sqrt{1-a_i}}$

$S_1 = \sum \frac{1}{\sqrt{1-a_i}}$

$S_2 = \sum \sqrt{1-a_i}$

$$S_2 = \sum_{i=1}^{n}\left(\sqrt{1-a_i} \times 1\right)$$

$$\leq \left(\left(\sum_{i=1}^{n}\left(\sqrt{1-a_i}\right)^2\right)\left(\sum_{i=1}^{n}(1^2)\right)\right)^{\frac{1}{2}} \quad \text{... using Cauchy Schwarz}$$

$$= \left(\sum_{i=1}^{n}(1-a_i)\, n\right)^{\frac{1}{2}}$$

$$= \left(n\left(n-\sum_{i=1}^{n}a_i\right)\right)^{\frac{1}{2}}$$

$$= \left(n(n-1)\right)^{\frac{1}{2}}$$

$\sum_{i=1}^{n} c = cn$

$\underbrace{c+c+c+\dots+c}_{n \text{ times}} = \sum_{i=1}^{n} c$

$$S_2 \leq \left(n(n-1)\right)^{\frac{1}{2}}$$

$$S \geq n\left(\frac{n}{n-1}\right)^{\frac{1}{2}} - \left(n(n-1)\right)^{\frac{1}{2}} = \frac{1}{\sqrt{n-1}}\left(n n^{\frac{1}{2}} - n^{\frac{1}{2}}(n-1)\right)$$

$$= \frac{1}{\sqrt{n-1}}\left(n n^{\frac{1}{2}} - n n^{\frac{1}{2}} + n^{\frac{1}{2}}\right)$$

$$= \frac{1}{\sqrt{n-1}}\sqrt{n}$$

$$\Rightarrow S \geq \frac{\sqrt{n}}{\sqrt{n-1}} \quad \text{—} (1)$$

$$\Rightarrow \quad \cdots \geq \frac{\quad}{\sqrt{n-1}}$$

$$\sum_{i=1}^{n} (\sqrt{a_i} \times 1) \leq \left(\sum_{i=1}^{n} (\sqrt{a_i})^2 \sum_{i=1}^{n} 1^2 \right)^{\frac{1}{2}} = \left(\left(\sum_{i=1}^{n} a_i \right) n \right)^{\frac{1}{2}} = n^{\frac{1}{2}}$$

$$\Rightarrow \sum_{i=1}^{n} \sqrt{a_i} \leq \sqrt{n}$$

From (i) we get, $S \geq \frac{\sqrt{n}}{\sqrt{n-1}} \geq \frac{\sum_{i=1}^{n} \sqrt{a_i}}{\sqrt{n-1}}$

Q> $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n \in \mathbb{R}$ such that,

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2 = 1$$

Prove that,

$$(a_1 b_2 - a_2 b_1)^2 \leq 2 \left| a_1 b_1 + a_2 b_2 + \cdots + a_n b_n - 1 \right|$$

Ans:- $\left(\sum a_i^2\right)\left(\sum b_i^2\right) \geq \left(\sum a_i b_i\right)^2$

$$\Rightarrow \left(\sum a_i b_i\right)^2 \leq 1$$

$$\Rightarrow -1 \leq \sum a_i b_i \leq 1 \qquad \Rightarrow -2 \leq \sum a_i b_i - 1 \leq 0$$

$$\Rightarrow \left| \sum a_i b_i - 1 \right| \leq 2$$

$$\left(\underbrace{\sum_{i=1}^{n} a_i^2}_{1}\right)\left(\underbrace{\sum_{i=1}^{n} b_i^2}_{1}\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{i,j=1}^{n} (a_i b_j - a_j b_i)^2 \geq (a_1 b_2 - b_1 a_2)^2$$

$$1 - \left(\sum a_i b_i\right)^2 \geq (a_1 b_2 - b_2 a_1)^2$$

$x^2 \leq b$

$x^2 \leq |b|$

$$\Rightarrow \left(\underline{1 + \sum a_i b_i}\right)\left(\underline{1 - \sum a_i b_i}\right) \geq (a_1 b_2 - b_2 a_1)^2$$

In total it is true

$-1 \leq \sum a_i b_i \leq 1$

$$\Rightarrow (a_1 b_2 - b_2 a_1)^2 \leq \left| \left(1 + \sum a_i b_i\right)\left(1 - \sum a_i b_i\right) \right|$$

$2 \leq \sum a_i b_i - 1 \leq 0$

$0 \leq \sum a_i b_i + 1 \leq 2$

$$= \left| 1 + \sum a_i b_i \right| \left| \sum a_i b_i - 1 \right|$$

$\left| \sum a_i b_i + 1 \right| \leq 2$

$\left| \sum a_i b_i - 1 \right| \leq 2$

$|ab| = |a||b|$

$$\leq 2 \left| \sum a_i b_i - 1 \right|$$

$|a - b| = |b - a|$

$|ab| = |a||b|$ $\leq 2\left|\Sigma a_i b_i - 1\right|$ $|\Sigma a_i b_i - 1| \leq 2$

$|a-b| = |b-a|$

Schwarz Inequality :-

$a_1, a_2, \dots, a_n \in \mathbb{R}$ and $b_1, b_2, \dots, b_n \in \mathbb{R}$ and $b_i > 0 \ \forall i \in \{1, 2, \dots, n\}$

we have,

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \dots + a_n)^2}{b_1 + b_2 + \dots + b_n}$$

Q) Let a, b, c, d, e be non-negative real numbers such that $a+b+c+d+e = 5$. Prove that,

HomeWork $$abc + bcd + cde + dea + eab \leq 5$$

Ans:- Hint: $a \leq b \leq c \leq d \leq e$

Write the whole expression in terms of a